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THE *TR*-GROUPS OF THE SPHERE SPECTRUM AT THE PRIME TWO

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Abstract: For the multiplicative group S^1, the circle, we have the topological Hochschild S^1-spectrum $T(\mathbf{S})$ of the sphere spectrum $\mathbf{S}$. For the finite cyclic group C_r $(\subset S^1)$ of order r, the *TR-groups of* $\mathbf{S}$ *at* 2 are defined by the equivariant homotopy groups $TR_k^n(\mathbf{S};2) = [S^k \wedge (S^1/C_{2^{n-1}})_+, T(\mathbf{S})]_{S^1}$ for $k \geq 0$ and $n \geq 1$. By the "trace method", the groups are closely related with the algebraic K-groups of $\mathbf{S}$. In [1], Hesselholt determined the TR-groups for $0 \leq k \leq 5$, in order to obtain the homotopy groups of the topological Whitehead spectrum of the circle in dimensions less than 4. In this paper, we extend his result for the TR-groups to $k \leq 9$ by the mod 2 Adams spectral sequence as well as the Atiyah-Hirzebruch spectral sequence.

1. Introduction

Throughout this paper, we fix a prime $p = 2$ and denote by C_r the finite cyclic subgroup of the circle S^1 of order r. Let $T(X)$ denote the topological Hochschild homology spectrum of a ring spectrum X. Since $T(X)$ is an S^1-spectrum, we define the *TR-spectrum* $TR^n(X;2)$ *of level* n as the fixed point spectrum $T(X)^{C_{2^{n-1}}}$ for $n \geq 1$. The spectrum $TR(X;2)$ is given by

$$TR(X;2) = \operatorname{holim}_n TR^n(X;2),$$

the homotopy limit of the system $\{R : TR^n(X;2) \to TR^{n-1}(X;2)\}_n$ of the restriction maps. The Frobenius maps $F : TR^n(X;2) \to TR^{n-1}(X;2)$ induce a map $F : TR(X;2) \to TR(X;2)$, and $TC(X;2)$ is a spectrum fitting in the cofiber sequence

$$TC(X;2) \xrightarrow{i} TR(X;2) \xrightarrow{id-F} TR(X;2) \xrightarrow{\partial} \Sigma TC(X;2).$$

Consider the algebraic K-theory spectrum $K(X)$ of a ring spectrum X, and the cyclotomic trace map $trc : K(X) \to TC(X;2)$. The "trace method" is to study $K(X)$ through the composite

$$tr_n : K(X) \xrightarrow{trc} TC(X;2) \xrightarrow{i} TR(X;2) \to TR^n(X;2).$$

We call the homotopy groups $TR_*^n(X;2) = \pi_*(TR^n(X;2))$ the (*2-primary*) *TR-groups of* X *of level* n.

Let $\mathbf{S}$ denote the sphere spectrum localized at the prime two. In this paper, we consider the TR-groups $TR_*^n(\mathbf{S};2)$. We have the Segal-tom Dieck splitting $TR_*^n(\mathbf{S};2) \cong \pi_*^S((BC_{2^{n-1}})_+) \oplus TR_*^{n-1}(\mathbf{S};2)$ ([1, p. 137, p. 148, p. 155]), where $BC_{2^{n-1}}$ denotes the classifying space of $C_{2^{n-1}}$. By definition, $TR_*^1(\mathbf{S};2) = \pi_*(T(\mathbf{S}))$, which is isomorphic to $\pi_*(\mathbf{S})$ ([1, p. 147]). These show an isomorphism

$$TR_*^n(\mathbf{S};2) \cong \pi_*(\mathbf{S}) \oplus \bigoplus_{1\le k<n} \pi_*^S((BC_{2^k})_+). \tag{1.1}$$

Hesselholt studied the Atiyah-Hirzebruch spectral sequence

$$E_{s,t}^2(n) = H_s(C_{2^n}, \pi_t(\mathbf{S})) \Rightarrow \pi_*^S((BC_{2^n})_+) \cong \pi_*(\mathbf{S}) \oplus \pi_*^S(BC_{2^n}), \tag{1.2}$$

which is called the skeleton spectral sequence in [1, p. 148], to show the following theorem.

THEOREM 1.3 (Hesselholt [1, Theorem 11]). *The TR-groups $TR_k^n(\mathbf{S};2)$ for $k \le 5$ are given by*

$$TR_0^n(\mathbf{S};2) \cong \mathbf{Z}_{(2)}^{\oplus n},$$

$$TR_1^n(\mathbf{S};2) \cong \mathbf{Z}/2^{\oplus n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^s,$$

$$TR_2^n(\mathbf{S};2) \cong \mathbf{Z}/2^{\oplus n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2,$$

$$TR_3^n(\mathbf{S};2) \cong \mathbf{Z}/8^{\oplus n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^{\max\{3,s+1\}} \oplus \bigoplus_{2\le s<n} \mathbf{Z}/2,$$

$$TR_4^n(\mathbf{S};2) \cong \bigoplus_{1\le s<n} \mathbf{Z}/2^{\min\{3,s\}},$$

$$TR_5^n(\mathbf{S};2) \cong \bigoplus_{2\le s<n} \mathbf{Z}/2^s \oplus \bigoplus_{3\le s<n} \mathbf{Z}/2.$$

Liulevicius determined the stable homotopy groups $\pi_k^S(BC_2)$ for $k \le 9$ ([3, Theorem II.6]). We consider $\pi_k^S(BC_{2^n})$ for $n > 1$ and $k \le 9$ in this paper. In section 2, we determine the stable homotopy group $\pi_6^S(BC_{2^n})$ by the Atiyah-Hirzebruch spectral sequence, and in section 3, we determine the stable homotopy groups $\pi_*^S(BC_{2^n})$ in dimensions 7, 8 and 9 by the mod 2 Adams spectral sequence as well as the results in section 2. The following theorem summarizes Corollary 2.10 and Propositions 3.12, 3.14 and 3.16.

THEOREM 1.4. *The TR-groups $TR_k^n(\mathbf{S};2)$ for $6 \le k \le 9$ are given by*

$$TR_6^n(\mathbf{S};2) \cong \mathbf{Z}/2^{\oplus n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2,$$

$$TR_7^n(\mathbf{S};2) \cong \mathbf{Z}/16^{\oplus n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2 \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^{\max\{4,s+2\}} \oplus \bigoplus_{2\le s<n} \mathbf{Z}/2,$$

$$TR_8^n(\mathbf{S};2) \cong \mathbf{Z}/2^{\oplus 2n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^{\min\{4,s\}} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^{\oplus 2},$$

$$TR_9^n(\mathbf{S};2) \cong \mathbf{Z}/2^{\oplus 3n} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^{\oplus 3} \oplus \bigoplus_{1\le s<n} \mathbf{Z}/2^{\min\{4,s\}} \oplus \bigoplus_{2\le s<n} \mathbf{Z}/2^{s-1}.$$

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2. The Atiyah-Hirzebruch spectral sequences

In this section, $E^r_{s,t}(n)$ denotes an E^r-term of the Atiyah-Hirzebruch spectral sequence (1.2), and $E^*(n)$ stands for the spectral sequence. Since the C_{2^n}-action on the homotopy groups $\pi_*(\mathbf{S})$ is trivial ([1, p. 145]), the standard resolution gives rise to isomorphisms

(2.1) $$E^2_{s,t}(n) = H_s(C_{2^n}, \pi_t(\mathbf{S})) \cong \begin{cases} \pi_t(\mathbf{S}) & s = 0, \\ \pi_t(\mathbf{S})/(2^n) & s : odd > 0, \\ \pi_t(\mathbf{S})[2^n] & s : even > 0, \end{cases}$$

of groups, where $\pi_t(\mathbf{S})[2^n]$ denotes the kernel of $\pi_t(\mathbf{S}) \xrightarrow{2^n} \pi_t(\mathbf{S})$.

Theorem 2.2 (*cf.* Toda [5, p. 189–190]). *The homotopy groups $\pi_k(\mathbf{S})$ for $k \leq 10$ are given by*

k	0	1	2	3	4	5	6	7	8	9	10
$\pi_k(\mathbf{S})$	$\mathbf{Z}_{(2)}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/8$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/16$	$\mathbf{Z}/2^{\oplus 2}$	$\mathbf{Z}/2^{\oplus 3}$	$\mathbf{Z}/2$
gen.	ι	η	η^2	ν			ν^2	σ	$\eta\sigma$, ε	$\eta\varepsilon$, μ, ν^3	$\eta\mu$

The generators satisfy the relations $\eta^3 = 4\nu$, $\eta^2\sigma = \eta\varepsilon + \nu^3$ and $\nu\sigma = 0$.

We notice that the spectral sequence (1.2) splits into the direct sum of two spectral sequences

$$E^2_{0,*}(n) \Rightarrow \pi_*(\mathbf{S}) \qquad \text{and} \qquad \bigoplus_{s\geq 1} E^2_{s,*}(n) \Rightarrow \pi^S_*(BC_{2^n})$$

([1, p. 148]). We study the latter spectral sequence.

First we consider the case for $n = 1$. By (2.1) and Theorem 2.2, the E^2-terms $E^2_{s,t}(1)$ for $s \geq 1$ and $s + t \leq 10$ are given by

s \ $s+t$	1	2	3	4	5	6	7	8	9	10
10										0
9									$\mathbf{Z}/2$	$\mathbf{Z}/2$
8								0	$\mathbf{Z}/2$	$\mathbf{Z}/2$
7							$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
6						0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$4\mathbf{Z}/8$	0
5					$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	0
4				0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$4\mathbf{Z}/8$	0	0	$\mathbf{Z}/2$
3			$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$
2		0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\boxed{4\mathbf{Z}/8}$	0	0	$\mathbf{Z}/2$	$8\mathbf{Z}/16$	$\mathbf{Z}/2^{\oplus 2}$
1	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2^{\oplus 2}$	$\mathbf{Z}/2^{\oplus 3}$

Hereafter $2^a\mathbf{Z}/2^b$ denotes the subgroup of $\mathbf{Z}/2^b$ generated by 2^a, which is isomorphic to $\mathbf{Z}/2^{b-a}$ if $a < b$, and zero otherwise. For example, in the above chart, the boxed $4\mathbf{Z}/8$ at $(s, t) = (2, 3)$ is the subgroup of $\mathbf{Z}/8 \cdot v$ generated by $4v$.

We deduce

(2.3) $$(E^2_{s,t}(n) \xrightarrow{d^2} E^2_{s-2,t+1}(n)) = \begin{cases} \times\eta & 4 \leq s \equiv 0, 1 \bmod (4), \\ 0 & \textit{otherwise}, \end{cases}$$

from [1, p. 148]. This implies that the E^3-terms have a periodicity:

(2.4) *The E^3-term $E^3_{s,t}(n)$ is isomorphic to $E^3_{s+4,t}(n)$ if $s \geq 2$.*

We obtain the E^3-terms $E^3_{s,t}(1)$ for $s \geq 1$ and $s + t \leq 9$ as follows by (2.3) and (2.4).

s \ $s+t$	1	2	3	4	5	6	7	8	9
9									0
8								0	0
7							$\mathbf{Z}/2$	0	0
6						0	$\mathbf{Z}/2$	0	0
5					0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0
4				0	0	0	$4\mathbf{Z}/8$	0	0
3			$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$
2		0	$\mathbf{Z}/2$	0	0	0	0	$\mathbf{Z}/2$	$8\mathbf{Z}/16$
1	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2^{\oplus 2}$

THEOREM 2.5 (Liulevicius [3, Theorem II.6]). *The stable homotopy groups of $BC_2 = \mathbf{R}P^\infty$, the infinite real projective space, in dimensions less than* 10 *are given by*

k	1	2	3	4	5	6	7	8	9
$\pi_k^S(\mathbf{R}P^\infty)$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/8$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/16 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2^{\oplus 3}$	$\mathbf{Z}/2^{\oplus 4}$

COROLLARY 2.6. *The spectral sequence $E^*(1)$ collapses at E^3 for $s + t \leq 9$.*

We turn to the case for $n \geq 2$. By (2.3) and (2.4), we have the following chart of E^3-terms of $E^*(n)$ for $s \geq 1$ and $s + t \leq 10$:

s										
10										0
9									$2\mathbf{Z}/2^n$	0
8								0	0	0
7							$\mathbf{Z}/2^n$	0	0	$\mathbf{Z}/4$
6						0	$\boxed{\mathbf{Z}/2}$	0	$\tilde{K}_{3,n}$	0
5					$2\mathbf{Z}/2^n$	0	Z_n	$C_{3,n}$	0	0
4				0	0	0	$K_{3,n}$	0	0	$\boxed{\mathbf{Z}/2}$
3			$\mathbf{Z}/2^n$	0	0	$\boxed{\mathbf{Z}/4}$	0	0	$\mathbf{Z}/2$	$C_{7,n}$
2		0	$\mathbf{Z}/2$	0	$\tilde{K}_{3,n}$	0	0	$\mathbf{Z}/2$	$K_{7,n}$	$\tilde{K}_{8,n}$
1	$\mathbf{Z}/2^n$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$C_{3,n}$	0	0	$\mathbf{Z}/2$	$C_{7,n}$	$\boxed{\mathbf{Z}/2^{\oplus 2}}$	$\mathbf{Z}/2^{\oplus 3}$
	1	2	3	4	5	6	7	8	9	10 $\quad s+t$

Here, $K_{t,n} = \pi_t(\mathbf{S})[2^n]$, $\tilde{K}_{t,n} = K_{t,n}/(\eta)$, $C_{t,n} = \pi_t(\mathbf{S})/2^n$ and $Z_n = \ker(C_{2,n} \xrightarrow{\eta} C_{3,n})$, whose structures are:

$$K_{3,n} \cong 2^{\max\{3-n,0\}}\mathbf{Z}/8, \qquad K_{7,n} \cong 2^{\max\{4-n,0\}}\mathbf{Z}/16, \qquad \tilde{K}_{3,n} \cong 2^{\max\{3-n,0\}}\mathbf{Z}/4,$$

$$\tilde{K}_{8,n} \cong \mathbf{Z}/2 \text{ except for } \tilde{K}_{8,2} \cong \tilde{K}_{8,3} \cong \mathbf{Z}/2^{\oplus 2}, \qquad C_{3,n} \cong \mathbf{Z}/2^{\min\{n,3\}},$$

$$C_{7,n} \cong \mathbf{Z}/2^{\min\{n,4\}} \quad \text{and} \quad Z_n = 0 \text{ except for } Z_2 \cong \mathbf{Z}/2.$$

LEMMA 2.7 ([1, p. 145, Lemma 6, p. 148]). *The Verschiebung map $V : \pi_*^S((BC_{2^{n-1}})_+) \to \pi_*^S((BC_{2^n})_+)$ induces a map $V : E^*(n-1) \to E^*(n)$ of spectral sequences. Let $\{x\}_n$ denote an element of $E_{s,t}^2(n)$ represented by $x \in \pi_t(\mathbf{S})$. If s is even, then $V(\{x\}_{n-1}) = \{x\}_n$ for the map $V : E_{s,t}^2(n-1) \to E_{s,t}^2(n)$ of the E_2-terms.*

Since the differentials $E^3_{6,1}(1) \xrightarrow{d^3} E^3_{3,3}(1)$ and $E^3_{4,6}(1) \xrightarrow{d^3} E^3_{1,8}(1)$ are trivial by Corollary 2.6, the differentials $E^3_{6,1}(n) \xrightarrow{d^3} E^3_{3,3}(n)$ and $E^3_{4,6}(n) \xrightarrow{d^3} E^3_{1,8}(n)$ for $n \geq 2$ are trivial by Lemma 2.7.

Recall [1, Lemma 8] that

$$(2.8)\quad (E^4_{s,t}(n) \xrightarrow{d^4} E^4_{s-4,t+3}(n)) = \begin{cases} \times\nu & 4 < s \equiv 0,1,2,3,8,9,10,11 \bmod (16), \\ \times 2\nu & 4 < s \equiv 6,7,12,13 \bmod (16), \\ 0 & \textit{otherwise}, \end{cases}$$

for $n \geq 1$. We then obtain the following chart of the E^5-terms for $n \geq 2$, except for the underlined group $E^5_{7,3}(n)$.

s										
10										0
9									$\boxed{8\mathbf{Z}/2^n}$	0
8								0	0	0
7							$2\mathbf{Z}/2^n$	0	0	$\underline{0}$
6						0	$\mathbf{Z}/2$	0	$\boxed{\tilde{K}_{3,n}}$	0
5					$2\mathbf{Z}/2^n$	0	Z_n	$\mathbf{Z}/2$	0	0
4				0	0	0	$K_{3,n}$	0	0	?
3			$\mathbf{Z}/2^n$	0	0	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	?
2		0	$\mathbf{Z}/2$	0	$\tilde{K}_{3,n}$	0	0	$\boxed{\mathbf{Z}/2}$	$K_{7,n}$	?
1	$\mathbf{Z}/2^n$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$C_{3,n}$	0	0	$\mathbf{Z}/2$	$\boxed{C_{7,n}}$	$\mathbf{Z}/2^{\oplus 2}$	?
	1	2	3	4	5	6	7	8	9	10 $s+t$

By (2.1) and Theorem 2.2, we see that $E^4_{11,0}(n) = \mathbf{Z}/2^n \cdot \iota$, and that $E^4_{7,3}(n)$ is a quotient of $E^3_{7,3}(n) = \mathbf{Z}/4 \cdot \nu$. Thus, we deduce from (2.8) that the group $E^5_{7,3}(n)$ is zero.

Lemma 2.9. *On $E^r_{s,t}(n)$ for $n \geq 2$ and $r \geq 5$, the only possibly nonzero differentials are $E^5_{6,3}(n) \xrightarrow{d^5} E^5_{1,7}(n)$, $E^7_{9,0}(n) \xrightarrow{d^7} E^7_{2,6}(n)$ and $E^8_{9,0}(n) \xrightarrow{d^8} E^8_{1,7}(n)$ for $s+t \leq 10$.*

Corollary 2.10. *For $n \geq 2$, the stable homotopy groups $\pi^S_*(BC_{2^n})$ in dimensions from 6 to 9 satisfy the following relations:*

$$\pi^S_6(BC_{2^n}) \cong \mathbf{Z}/2,$$

$$|\pi^S_7(BC_{2^n})| = 2^{n+4},$$

$$|\pi_8^S(BC_{2^n})| \le 2^{\min\{n+2,6\}},$$
$$|\pi_9^S(BC_{2^n})| \le 2^{\min\{2n+2,n+6\}}.$$

3. The mod 2 Adams spectral sequence

In this section, we consider the mod 2 Adams spectral sequence

$$E_2^{s,t}(X) = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\tilde{H}^*(X), \mathbf{Z}/2) \Rightarrow \pi_{t-s}^S(X)$$

for a space X. Here $\tilde{H}^*(X)$ denotes a reduced cohomology of X with coefficients $\mathbf{Z}/2$, and $\mathscr{A}$ denotes the Steenrod algebra. We assume that $n \ge 2$, and determine the stable homotopy groups $\pi_*^S(BC_{2^n})$ in dimensions less than 10 by the mod 2 Adams spectral sequence for BC_{2^n}.

Proposition 3.1. *The E_2-term $E_2^{*,*}(BC_{2^n})$ is isomorphic to $xE_2^{*,*}(S^0) \oplus E_2^{*,*}(\mathbf{C}P^\infty) \oplus xE_2^{*,*}(\mathbf{C}P^\infty)$ as a graded $E_2^{*,*}(S^0)$-module for a generator $x \in E_2^{1,0}(BC_{2^n})$. Here S^0 and $\mathbf{C}P^\infty$ denote the 0-dimensional sphere and the infinite complex projective space, respectively.*

Proof. We claim that there exists a generator $x \in \tilde{H}^1(BC_{2^n})$ such that

$$\tilde{H}^*(BC_{2^n}) \cong \mathbf{Z}/2 \cdot x \oplus \tilde{H}^*(\mathbf{C}P^\infty) \oplus x\tilde{H}^*(\mathbf{C}P^\infty) \tag{3.2}$$

as a graded $\mathscr{A}$-algebra. Indeed, the unreduced cohomology $H^*(BC_{2^n}, \mathbf{Z}/2)$ is isomorphic to the group cohomology $H^*(C_{2^n}, \mathbf{Z}/2) \cong E(x) \otimes P(y)$ with $|x| = 1$ and $|y| = 2$. Here $E(-)$ and $P(-)$ denote the exterior and the polynomial algebras, respectively. Furthermore, we see that the action of $\mathscr{A}$ on the generators x and y is trivial except for $Sq^2(y) = y^2$ by the fundamental properties of the Steenrod squares, other than $Sq^1(y) = 0$. Note that Sq^1 fits in the exact sequence

$$H^1(BC_{2^n}, \mathbf{Z}/2) \xrightarrow{Sq^1} H^2(BC_{2^n}, \mathbf{Z}/2) \longrightarrow H^2(BC_{2^n}, \mathbf{Z}/4)$$
$$\longrightarrow H^2(BC_{2^n}, \mathbf{Z}/2) \xrightarrow{Sq^1} H^3(BC_{2^n}, \mathbf{Z}/2)$$

associated to the short exact sequence $0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0$. In the exact sequence, $H^2(BC_{2^n}, \mathbf{Z}/2^i) \cong \mathbf{Z}/2^i$ by the standard resolution. The first Sq^1 is zero, and so is the second Sq^1 as desired. We note that $\tilde{H}^*(S^0) \cong \mathbf{Z}/2$ and $\tilde{H}^*(\mathbf{C}P^\infty) \cong \bar{P}(y)$ as graded $\mathscr{A}$-algebras for the augmented ideal $\bar{P}(y)$ of $P(y)$. Thus, the claim (3.2) is verified and hence the proposition follows. □

The E_2-terms $E_2^{s,t}(S^0)$ are well known as follows ([4, Theorem 3.2.11]):

s \ $t-s$	0	1	2	3	4	5	6	7	8	9
$\vdots$	$\vdots$									
5	h_0^5									Ph_1
4	h_0^4							$h_0^3h_3$		h_1c_0
3	h_0^3			$h_1^3 = h_0^2h_2$				$h_0^2h_3$	c_0	$h_2^3 = h_1^2h_3$
2	h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3	
1	h_0	h_1		h_2				h_3		
0	1									

The generators satisfy the relations:

(3.3) $h_ih_{i+1} = 0$ for $i \geq 0$, $h_1^3 = h_0^2h_2$, $h_0h_2^2 = 0$, $h_2^3 = h_1^2h_3$,
$h_0^4h_3 = 0$, $h_0c_0 = 0$, $h_1^2c_0 = 0$ and $h_0Ph_1 = 0$.

We see the following fact immediately.

(3.4) *The* mod 2 *Adams spectral sequence for* S^0 *collapses at* E_2 *for* $t - s < 10$.

The E_2-terms $E_2^{s,t}(\mathbf{C}P^\infty)$ are determined in [3, Prop. II.3] as follows:

s \ $t-s$	0	1	2	3	4	5	6	7	8	9	10
$\vdots$			$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
5			$h_0^5e_2$		$h_0^4e_4$		$h_0^5e_6$		$h_0^2e_8$		$h_0^3e_{10}$
4			$h_0^4e_2$		$h_0^3e_4$		$h_0^4e_6$		h_0e_8	$h_0^3h_3e_2$	$h_0^2e_{10}$
3			$h_0^3e_2$		$h_0^2e_4$		$h_0^3e_6$		e_8	$h_0^2h_3e_2$	h_0e_{10}
2			$h_0^2e_2$		h_0e_4	$h_0h_2e_2$	$h_0^2e_6$		$h_1^2e_6$	$h_0h_3e_2$	e_{10}
1			h_0e_2		e_4	h_2e_2	h_0e_6	h_1e_6		h_3e_2	
0			e_2				e_6				

REMARK 3.5. In [3], the generators h_0, h_i $(i > 0)$, e_2, e_4, e_6, e_8 and e_{10} here are denoted by $g_0, h_{i-1}, e_{0,2}, e_{1,5}, e_{0,6}, e_{3,11}$ and $e_{2,12}$, respectively.

Therefore, we obtain the following chart of $E_2^{*,*}(BC_{2^n})$ by Proposition 3.1.

s \ $t-s$	0	1	2	3	4	5	6	7	8	9	10
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6		xh_0^6	$h_0^6e_2$	$xh_0^6e_2$	$h_0^5e_4$	$xh_0^5e_4$	$h_0^6e_6$	$xh_0^6e_6$	$h_0^3e_8$	$xh_0^3e_8$	$h_0^4e_{10}$
5		xh_0^5	$h_0^5e_2$	$xh_0^5e_2$	$h_0^4e_4$	$xh_0^4e_4$	$h_0^5e_6$	$xh_0^5e_6$	$h_0^2e_8$	$xh_0^2e_8$	xPh_1 $h_0^3e_{10}$
4		xh_0^4	$h_0^4e_2$	$xh_0^4e_2$	$h_0^3e_4$	$xh_0^3e_4$	$h_0^4e_6$	$xh_0^4e_6$	$xh_0^3h_3$ h_0e_8	xh_0e_8 $h_0^3h_3e_2$	xh_1c_0 $xh_0^3h_3e_2$ $h_0^2e_{10}$
3		xh_0^3	$h_0^3e_2$	$xh_0^3e_2$	$xh_0^2h_2$ $h_0^2e_4$	$xh_0^2e_4$	$h_0^3e_6$	$xh_0^3e_6$	$xh_0^2h_3$ e_8	xc_0 xe_8 $h_0^2h_3e_2$	xh_2^3 $xh_0^2h_3e_2$ h_0e_{10}
2		xh_0^2	$h_0^2e_2$	xh_1^2 $xh_0^2e_2$	xh_0h_2 h_0e_4	xh_0e_4 $h_0h_2e_2$	$xh_0h_2e_2$ $h_0^2e_6$	xh_2^2 $xh_0^2e_6$	xh_0h_3 $h_1^2e_6$	xh_1h_3 $xh_1^2e_6$ $h_0h_3e_2$	$xh_0h_3e_2$ e_{10}
1		xh_0	xh_1 h_0e_2	xh_0e_2	xh_2 e_4	xe_4 h_2e_2	xh_2e_2 h_0e_6	xh_0e_6 h_1e_6	xh_3 xh_1e_6	h_3e_2	xh_3e_2
0		x	e_2	xe_2			e_6	xe_6			

Recall a well known fact (*cf.* [4, Lemma 3.1.3]):

(3.6) *If* $\alpha \in \pi_*^S(BC_{2^n})$ *is detected by an element* a *in* $E_2^{*,*}(BC_{2^n})$, *then* 2α *is detected by* ah_0.

Since BC_{2^n} is a Hopf space (*cf.* [2]), the following holds (*cf.* [4, Theorem 2.3.3]).

(3.7) *The differentials of the* mod 2 *Adams spectral sequence for* BC_{2^n} *are derivations.*

By (1.1) and Theorem 2.2, the TR-groups in Theorem 1.3 give rise to the stable homotopy groups $\pi_k^S(BC_{2^n})$ for $k \leq 5$ as follows:

$$\begin{aligned}
&\pi_1^S(BC_{2^n}) \cong \mathbf{Z}/2^n,\\
&\pi_2^S(BC_{2^n}) \cong \mathbf{Z}/2,\\
(3.8)\qquad &\pi_3^S(BC_{2^n}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2^{n+1},\\
&\pi_4^S(BC_{2^n}) \cong \mathbf{Z}/2^{\min\{3,n\}},\\
&\pi_5^S(BC_{2^n}) \cong \mathbf{Z}/2^{\oplus \min\{1,n-2\}} \oplus \mathbf{Z}/2^n.
\end{aligned}$$

We obtain the following lemma from (3.8).

Lemma 3.9. *In the* mod 2 *Adams spectral sequence for* BC_{2^n}, *the elements* x, xe_2 *and* xe_4 *are permanent cycles,*

$$d_n(e_2) = xh_0^n, \qquad d_n(e_4) = xh_0^{n+1}e_2 \qquad \text{and} \qquad d_2(e_6) = \begin{cases} h_0h_2e_2 + xh_0e_4 & n = 2,\\ h_0h_2e_2 & n > 2.\end{cases}$$

Furthermore, $d_2(h_2e_2) = xh_0^2h_2$ *if* $n = 2$, *and* h_2e_2 *is a permanent cycle otherwise.*

Lemma 3.10. *The elements* h_1e_6 *and* xh_0e_6 *of* $E_2^{1,8}(BC_{2^n})$ *are permanent cycles.*

Proof. We note that $\pi_6^S(BC_{2^n}) \cong \mathbf{Z}/2$ by Corollary 2.10. Since xh_2e_2 is a permanent cycle by Lemma 3.9, it detects a generator of $\pi_6^S(BC_{2^n})$, and so h_0e_6 supports a nonzero differential. We deduce $d_n(h_0e_6) = xh_0^ne_4$ from the structure of $\pi_5^S(BC_{2^n})$ in (3.8). Therefore $h_0^ie_6$ for $i \geq 1$ cannot be a target of any differential. □

Lemma 3.11. $d_n(e_8) = xh_0^{n+3}e_6$.

Proof. By (3.4), (3.7), Lemmas 3.9 and 3.10, the elements xh_0e_6, h_1e_6 and xh_2^2 (resp. xh_1e_6, xh_3 and $h_1^2e_6$) detect generators of $\pi_7^S(BC_{2^n})$ (resp. $\pi_8^S(BC_{2^n})$). Since $|\pi_7^S(BC_{2^n})| = 2^{n+4}$ by Corollary 2.10, and the elements h_1e_6 and xh_2^2 generate the $\mathbf{Z}/2$-summands, the element detected by $xh_0^{n+3}e_6$ is zero in the homotopy. □

Proposition 3.12. $\pi_7^S(BC_{2^n}) \cong \mathbf{Z}/2^{\oplus 2} \oplus \mathbf{Z}/2^{n+2}$. *The generators of summands are detected by* xh_2^2, h_1e_6 *and* xh_0e_6, *respectively.*

Lemma 3.13. *The element* $h_3e_2 \in E_2^{1,10}(BC_{2^n})$ *is a permanent cycle if* $n > 3$, *and* $d_n(h_3e_2) = xh_0^nh_3$ *if* $n = 2, 3$. *The element* xe_8 *is a permanent cycle.*

Proof. Since $d_n(e_2) = xh_0^n$ by Lemma 3.9, we have $d_n(h_3e_2) = xh_0^nh_3$ by (3.7), which is not zero if $n = 2, 3$, and zero if $n > 3$. By (3.7) and Lemma 3.11, $h_0^ie_8$ supports a nontrivial differential, and so it cannot be a target of an Adams differential. Therefore $d_r(h_3e_2) = 0$ for $r > n$ in the case for $n > 3$.

Since $d_n(xe_8)=0$ by Lemma 3.11, we see that $d_r(xe_8)=0$ for $r>n$ similarly. □

This together with Lemma 3.10 implies the following result.

PROPOSITION 3.14. *$\pi_8^S(BC_{2^n})\cong \mathbf{Z}/2^{\oplus 2}\oplus \mathbf{Z}/2^{\min\{n,4\}}$. The generators of summands are detected by xh_1e_6, $h_1^2e_6$ and xh_3, respectively.*

LEMMA 3.15. $|\pi_9^S(BC_{2^n})|=2^{\min\{2n+2,n+6\}}$.

PROOF. Proposition 3.14 shows that $|\pi_8^S(BC_{2^n})|=2^{\min\{n+2,6\}}$, which implies that the undetermined differentials in Lemma 2.9 turn out to be trivial. We now see the lemma by the same argument as the proof of Corollary 2.10. □

PROPOSITION 3.16. *$\pi_9^S(BC_{2^n})\cong \mathbf{Z}/2^{\oplus 3}\oplus \mathbf{Z}/2^{\min\{n,4\}}\oplus \mathbf{Z}/2^{n-1}$. The generators of summands are detected by xc_0, $xh_1^2e_6$, xh_1h_3, $h_0^{\max\{4-n,0\}}h_3e_2$ and xe_8, respectively.*

PROOF. Since $d_2(xh_3e_2)=0$ by Lemma 3.13, we see that xc_0 and xh_1h_3 generate $\mathbf{Z}/2$-summands by (3.4) and (3.7). The element $xh_1^2e_6$ detects a generator of the other $\mathbf{Z}/2$ summand by Lemma 3.10. Lemma 3.13 shows that $h_0^{\max\{4-n,0\}}h_3e_2$ generates the summand $\mathbf{Z}/2^{\min\{n,4\}}$. Lemmas 3.13 and 3.15 imply that xe_8 generates the summand $\mathbf{Z}/2^{n-1}$. □

REMARK 3.17. This also implies a differential $d_n(e_{10})=xh_0^{n-1}e_8$ for $n>2$, and $d_2(e_{10})\equiv xh_0e_8 \bmod (h_0^3h_3e_2)$ for $n=2$.

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